MODELE MATEMATYCZNE W UBEZPIECZENIACH

POMOCNICZE WZORY

$$i = \left(1 + \frac{i^{(m)}}{m}\right)^m - 1,$$

$$k(t) = k(0)\mathrm{e}^{\delta t}, \quad \text{gdzie } \delta = \log(1+i)$$

$$d = \frac{i}{i+1}$$

$$d^{(m)} = \frac{i^{(m)}}{1 + \frac{i^{(m)}}{m}}$$

- $$\ddot{a}_{\overline{\infty}|} = \frac{1}{d}.$$
- $$a_{\overline{\infty}|} = \frac{1}{i}.$$
- $$\ddot{a}^{(m)}_{\overline{\infty}|} = \frac{1}{d^{(m)}}$$
- $$a^{(m)}_{\overline{\infty}|} = \frac{1}{i^{(m)}}$$
- $$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d};$$
- $$a_{\overline{n}|} = \frac{1-v^n}{i};$$
- $$\ddot{a}^{(m)}_{\overline{n}|} = \frac{1-v^n}{d^{(m)}};$$
- $$a^{(m)}_{\overline{n}|} = \frac{1-v^n}{i^{(m)}}$$

$$ {}_{s|t}q_x = {}_{s+t}q_x - {}_s q_x = {}_s p_x - {}_{s+t}p_x; $$

$$ {}_t p_{[x]+s} = \frac{{}_{s+t}p_x}{{}_s p_x}, $$

$$ {}_t q_{[x]+s} = \frac{{}_{s|t}q_x}{{}_s p_x} $$

$$ {}_{s+t}p_x = {}_s p_x \, {}_t p_{[x]+s} $$

$$ {}_{s|t}q_x = {}_s p_x \, {}_t q_{[x]+s}, $$

$$ {}_k p_x = p_x \prod_{i=1}^{k-1} p_{[x]+i}. $$

$$ \mathring{e}_x = \int_0^\infty {}_t p_x dt. $$

$$ \mu_{[x]+t} = \frac{f_x(t)}{1 - F_x(t)} = -\frac{1}{{}_t p_x} \frac{d({}_t p_x)}{dt} $$

$$ \mathrm{P}(K_x = k) = {}_{k|1}q_x $$

$$ e_x = \sum_{k=1}^{\infty} {}_k p_x. $$

- Rozkład de Moivre'a

$$ \mu_t = \frac{1}{\omega - t}, \quad 0 \leqslant t \leqslant \omega. $$

- Rozkład Gompertza

$$ \mu_t = Bc^t, \quad t > 0, \quad \text{gdzie } B > 0 \text{ i } c > 1. $$

- Rozkład Makehama

$$ \mu_t = A + Bc^t, \quad t > 0, \quad \text{gdzie } B > 0 \text{ i } c > 1 \text{ oraz } A \geqslant -B. $$

- Rozkład Weibulla

$$ \mu_t = kt^n, \quad t \geqslant 0, \quad \text{gdzie } k > 0,\ n > 0. $$

HJP

$$ {}_t p_x = \frac{{}_{x+t}p_0}{{}_x p_0} $$

$$ {}_t p_{[x]+u} = {}_t p_{x+u} \qquad (*) $$

$$\mu_{[x]+t} = \mu_{x+t}. \tag{**}$$

$$_tp_x = \exp\left(-\int_x^{x+t} \mu_u du\right)$$

HA

$$_kp_x = \frac{_{x+k}p_0}{_xp_0}$$

$$p_{[x]+k} = p_{x+k}$$

$$_kp_x = p_x p_{x+1} \dots p_{x+k-1}.$$

HU

$$_{n+u}p_x = (1-u)_np_x + u \cdot {}_{n+1}p_x, \quad 0 \leqslant u < 1,\ n = 0, 1, 2, \dots .$$

$$_up_x = 1 - uq_x, \quad _uq_x = uq_x.$$

Ubezpieczenia płatne w chwili śmierci.

$$\bar{A}_x = \int_0^\infty v^t {}_tp_x \mu_{x+t} dt.$$

$$\bar{A}^1_{x:\overline{n}|} = \int_0^\infty v^t {}_tp_x \mu_{x+t} dt.$$

$$\bar{A}_{x:\overset{1}{\overline{n}|}} = v^n {}_np_x.$$

$$\bar{A}_{x:\overline{n}|} = \bar{A}^1_{x:\overline{n}|} + \bar{A}_{x:\overset{1}{\overline{n}|}}.$$

$$_{m|}\bar{A}_x = {}_mp_x v^m \bar{A}_{x+m}.$$

Ubezpieczenia płatne na koniec roku śmierci.

$$A_x = \sum_{k=0}^\infty v^{k+1} {}_kp_x q_{x+k}.$$

$$A^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} {}_kp_x q_{x+k}.$$

$$A_{x:\overset{1}{\overline{n}|}} = v^n {}_np_x.$$

$$A_{x:\overline{n}|} = A^1_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}},$$

$$_{m|}A_x = {}_mp_x v^m A_{x+m}.$$

$$\bar{A}_x = \frac{i}{\delta} A_x.$$

$$\bar{A}^1_{x:\overline{n}|} = \frac{i}{\delta} A^1_{x:\overline{n}|}.$$

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x.$$

Renty życiowe.

$$\ddot{a}_x = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|}\, {}_kp_x q_{x+k} = \sum_{k=0}^{\infty} v^k\, {}_kp_x = \frac{1 - A_x}{d}.$$

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|}\, {}_kp_x q_{x+k} = \sum_{k=0}^{n-1} v^k\, {}_kp_x = \frac{1 - A_{x:\overline{n}|}}{d}.$$

$${}_{m|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{m}|} = \sum_{k=m}^{\infty} v^k\, {}_kp_x = {}_mp_x v^m \ddot{a}_{x+m}.$$

$$\ddot{a}_x^{(m)} = \alpha(m)\ddot{a}_x - \beta(m),$$

gdzie

$$\alpha(m) = \frac{di}{d^{(m)} i^{(m)}} \approx 1, \quad \beta(m) = \frac{i - i^{(m)}}{d^{(m)} i^{(m)}} \approx \frac{m-1}{2m}$$

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \alpha(m)\ddot{a}_{x:\overline{n}|} - \beta(m)(1 - v^n\, {}_np_x).$$

Składki.

$$P_x = \frac{A_x}{\ddot{a}_x}.$$

$$P^1_{x:\overline{n}|} = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}.$$

$$P_{x:\overset{1}{\overline{n}|}} = \frac{A_{x:\overset{1}{\overline{n}|}}}{\ddot{a}_{x:\overline{n}|}}.$$

$$P_{x:\overline{n}|} = P^1_{x:\overline{n}|} + P_{x:\overset{1}{\overline{n}|}}.$$

$$P_x^{(m)} = \frac{A_x}{\ddot{a}_x^{(m)}}.$$

$$P_{x:\overset{1}{\overline{n}|}}^{(m)} = \frac{A_{x:\overset{1}{\overline{n}|}}}{\ddot{a}_{x:\overline{n}|}^{(m)}}.$$

$$P^{1\ (m)}_{x:\overline{n}|} = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}^{(m)}}.$$

$$P_{x:\overline{n}|}^{(m)} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}^{(m)}}.$$

$$ {}_hP_x = \frac{A_x}{\ddot{a}_{x:\overline{h}|}}. $$

$$ {}_hP^1_{x:\overline{n}|} = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{h}|}}; $$

$$ {}_hP_{x:\overline{n}|}^{\;\;1} = \frac{A_{x:\overline{n}|}^{\;\;1}}{\ddot{a}_{x:\overline{h}|}}; $$

$$ {}_hP_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{h}|}}. $$

Funkcje komutacyjne.

$$ D_x = v^x l_x, $$

$$ C_x = v^{x+1} d_x $$

$$ M_x = \sum_{k=0}^{\infty} C_{x+k} $$

$$ N_x = \sum_{k=0}^{\infty} D_{x+k} $$

$$ A_x = \frac{M_x}{D_x} $$

$$ A^1_{x:\overline{n}|} = \frac{M_x - M_{x+n}}{D_x} $$

$$ A_{x:\overline{n}|}^{\;\;1} = \frac{D_{x+k}}{D_x} $$

$$ A_{x:\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+k}}{D_x} $$

$$ \ddot{a}_x = \frac{N_x}{D_x}, $$

$$ \ddot{a}_{x:\overline{n}|} = \frac{N_x - N_{x+n}}{D_x}, $$

$$ {}_{n|}\ddot{a}_x = \frac{N_{x+n}}{D_x} $$

$$ P_x = \frac{M_x}{N_x}, $$

$$ P^1_{x:\overline{n}|} = \frac{M_x - M_{x+n}}{N_x - N_{x+n}}, $$

$$ P_{x:\overline{n}|}^{\;\;1} = \frac{D_{x+n}}{N_x - N_{x+n}}, $$

$$ P_{x:\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}. $$

Składki brutto.

$$\text{OWA składki brutto} = \text{OWA przyszłego świadczenia} + \text{OWA kosztów}$$

$$P^{\mathrm{br}}_{x:\overline{n}|} = \frac{A_{x:\overline{n}|} + \alpha + \gamma \ddot{a}_{x:\overline{n}|}}{(1-\beta)\ddot{a}_{x:\overline{n}|}} = \frac{1+\alpha}{1-\beta} P_{x:\overline{n}|} + \frac{\alpha d + \gamma}{1-\beta}$$

$${}_mP^{\mathrm{br}}_{x:\overline{n}|} = \frac{1+\alpha}{1-\beta} {}_mP_{x:\overline{n}|} + \frac{\alpha d + \gamma}{1-\beta} \cdot \frac{\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}}.$$

Rozkład dwupunktowy.

$$\mathrm{P}(N=1) = p, \quad \mathrm{P}(N=0) = q = 1-p,$$

$$EN = p, \quad \operatorname{Var} N = pq.$$

Rozkład dwumianowy.

$$P(N=k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, 2, \ldots, n,$$

$$EN = np, \quad \operatorname{Var} N = npq.$$

Rozkład Poissona.

$$\mathrm{P}(N=k) = \mathrm{e}^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots,$$

$$EN = \lambda, \quad \operatorname{Var} N = \lambda.$$

Rozkład ujemnie dwumianowy.

$$\mathrm{P}(N=k) = \binom{n+k-1}{k} p^n q^k, \quad k = 0, 1, 2, \ldots .$$

$$EN = \frac{nq}{p}, \quad \operatorname{Var} N = \frac{nq}{p^2},$$

Rozkład geometryczny.

$$P(N=k) = pq^k, \quad k = 0, 1, 2, \ldots,$$

$$EN = \frac{q}{p}, \quad \operatorname{Var} N = \frac{q}{p^2}.$$

Rozkład logarytmiczny.

$$\mathrm{P}(N=k) = -\frac{p^k}{k \log q}, \quad k = 1, 2, \ldots,$$

$$EN = -\frac{p}{q \log q}, \quad \operatorname{Var} N = \frac{p}{q(\log q)^2}(-\log q - p)$$

Model akumulacyjny.

$$q_n^{*k} = \mathrm{P}(S_k = n) = \sum_{i=1}^{n-1} q_i^{*(k-1)} q_{n-i}.$$

$$\mathrm{P}(N=n) = \sum_{k=0}^{n} p_k q_n^{*k}.$$

Jednorodny proces Poissona.

(a) $N_0 = 0$;

(b) dla $0 \leqslant s < t$ zmienne losowe N_s oraz $N_t - N_s$ są niezależne;

(c) dla $0 \leqslant s < t$

$$\mathrm{P}\{N_t - N_s = k\} = \mathrm{e}^{-\lambda(t-s)} \frac{(\lambda(t-s))^k}{k!} \quad k = 0, 1, 2, \ldots .$$

$$\mathrm{P}\{N_t = k\} = \mathrm{e}^{-\lambda t} \frac{(\lambda t)^k}{k!} \quad k = 0, 1, 2, \ldots .$$

$$EN_t = \lambda t = \mathrm{Var}\, N_t.$$

$$\mathrm{P}(T_k > t) = \mathrm{e}^{-\lambda t}, \quad t > 0.$$

Niejednorodny proces Poissona. Dla $0 \leqslant s < t$

$$\mathrm{P}\{N_t - N_s = k\} = \mathrm{e}^{-[m(t)-m(s)]} \frac{[m(t) - m(s)]^k}{k!} \quad k = 0, 1, 2, \ldots ,$$

$$\mathrm{P}\{N_t = k\} = \mathrm{e}^{-m(t)} \frac{[m(t)]^k}{k!} \quad k = 0, 1, 2, \ldots ,$$

Mieszany proces Poissona.

$$\mathrm{P}\{N_{s+t} - N_s = n\} = \int_0^\infty \mathrm{e}^{-\lambda t} \frac{(\lambda h)^n}{n!} dG(\lambda).$$

Jeżeli $\mathrm{P}(L = \lambda_i) = g_i, \quad i = 1, 2, \ldots,$ to

$$\mathrm{P}\{N_{s+t} - N_s = n\} = \sum_{i=0}^{\infty} \mathrm{e}^{-\lambda_i t} \frac{(\lambda_i h)^n}{n!} g_i.$$

Jeżeli L ma gęstość g, to

$$\mathrm{P}\{N_{s+t} - N_s = n\} = \int_0^\infty \mathrm{e}^{-\lambda t} \frac{(\lambda h)^n}{n!} g(\lambda) d\lambda.$$

$$\mathrm{P}\{L \leqslant x \mid N_t = n\} = \frac{\int_0^x \mathrm{e}^{-\lambda t} (\lambda h)^n dG(\lambda)}{\int_0^\infty \mathrm{e}^{-\lambda t} (\lambda h)^n dG(\lambda)}.$$

Rozkład wykładniczy.

$$f_X(x) = \begin{cases} \lambda \mathrm{e}^{-\lambda x}, & \text{dla } x > 0, \\ 0, & \text{dla } x \leqslant 0. \end{cases}$$

$$F_X(x) = \mathrm{P}(X \leqslant x) = \begin{cases} 1 - \mathrm{e}^{-\lambda x}, & \text{dla } x > 0, \\ 0, & \text{dla } x \leqslant 0. \end{cases}$$

$$EX = \frac{1}{\lambda}, \quad \mathrm{Var}\, X = \frac{1}{\lambda^2}, \quad M_X(t) = \frac{\lambda}{\lambda - t}.$$

Funkcja gamma.

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}dx, \quad \alpha > 0.$$

$$\Gamma(n+1) = n! = 1\cdot 2\cdot \ldots \cdot n, \quad n = 0, 1, 2, \ldots$$

$$\int_0^\infty x^{\alpha-1}\mathrm{e}^{-\beta x}dx = \frac{\Gamma(\alpha)}{\beta^\alpha}.$$

Rozkład gamma $\Gamma(\alpha,\beta)$.

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\mathrm{e}^{-\beta x}, \quad x > 0.$$

$$EX = \frac{\alpha}{\beta}, \quad \mathrm{Var} = \frac{\alpha}{\beta^2}, \quad M_X(t) = \left(\frac{\beta}{\beta - t}\right)^\alpha$$

Rozkład Pareto.

$$f_X(x) = \frac{\alpha}{(1+x)^{\alpha+1}}, \quad x \geqslant 0.$$

$$F_X(x) = 1 - \frac{1}{(1+x)^\alpha}, \quad x \geqslant 0.$$

$$EX = \frac{1}{\alpha - 1}, \quad \text{jeśli } \alpha > 1,$$

$$EX^2 = \frac{2}{(\alpha-1)(\alpha-2)}, \quad \text{jeśli } \alpha > 2.$$

Rozkład normalny.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in \mathbb{R}.$$

$$EX = \mu, \quad \mathrm{Var}\, X = \sigma^2, \quad M_X(t) = \mathrm{e}^{t\mu + \frac{\sigma^2 t^2}{2}}$$

Centralne twierdzenie graniczne.

$$\mathrm{P}\left(a \leqslant \frac{S_n - n\mu}{\sigma\sqrt{n}} \leqslant b\right) \sim \Phi(b) - \Phi(a).$$

Sploty rozkładów.

$$F_{X+Y}(x) = \int_0^x F_Y(x-s)f_X(s)ds$$

$$f_{X+Y}(x) = \int_0^x f_X(x-s)f_Y(s)ds$$

$$p_{X+Y}(n) = \mathrm{P}(X+Y=n) = \sum_{k=0}^n \mathrm{P}(X=k)\mathrm{P}(Y=n-k)$$

$$F_X * F_Y * F_Z = (F_X * F_Y) * F_Z$$

Funkcje tworzące momenty.

Dla zmiennej losowej dyskretnej

$$M_X(t) = \sum_k \mathrm{e}^{tx_k} \mathrm{P}(X = x_k).$$

Dla zmiennej losowej ciągłej

$$M_X(t) = \int_{-\infty}^{\infty} \mathrm{e}^{tx} f_X(x)\, dx.$$

- $M_X(0) = 1$
- $M_Y(t) = \mathrm{e}^{bt} M_X(at)$,
- Jeżeli $EX^n < \infty$, to $EX^n = M_X^{(n)}(0)$.
- Jeżeli X i Y są niezależne, to $M_{X+Y}(t) = M_X(t) M_Y(t)$.

Funkcje tworzące prawdopodobieństwa.

$$G_X(z) = Es^X = \sum_{k=0}^{\infty} p_k z^k, \quad |z| < 1.$$

- $G_X(1) = 1$ oraz $G_X(0) = \mathrm{P}(X = 0)$;
- $\mathrm{P}(X = k) = \frac{1}{k!} G_X^{(k)}(0)$
- $G_X^{(n)}(1) = E\Big[X(X-1)\dots(X-k+1)\Big]$
- $EX = G_X'(1)$ oraz $EX^2 = G_X''(1) + G_X'(1)$
- Jeżeli X i Y są niezależne, to $G_{X+Y}(z) = G_X(z) G_Y(z)$.

Model łącznego ryzyka.

$$Z_t = X_1 + X_2 + \dots + X_{N_t}.$$

$$M_{Z_t}(u) = G_{N_t}\left(M_X(u)\right),$$

$$E(Z_t) = \mu_1 E N_t$$

$$\mathrm{Var}(Z_t) = E(N_t)\, \mathrm{Var}(X_1) + \mu_1^2\, \mathrm{Var}(N_t).$$